

Is Quantization Geometry?*

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Abstract

The metric known to be relevant for standard quantization procedures receives a natural interpretation and its explicit use simultaneously gives both physical and mathematical meaning to a (coherent-state) phase-space path integral, and at the same time establishes a fully satisfactory, geometric procedure of quantization.

Introduction and Background

Purpose and achievements

The goal of the present work is to motivate and present a *conceptually simple, fully geometric prescription for quantization*. Such a goal has been, and continues to be, the subject of a number of research efforts. In the present

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paper we shall examine the issues from the point of view of coherent states. It will be our conclusion that the symplectic geometry of classical mechanics, augmented by a natural, and even necessary, metric on the classical phase space, are the only essential ingredients to provide a quantization scheme that is fully geometric in character and one that can be expressed in a coordinate-free form. Moreover, and in a surprising sense, we shall see that coherent states—far from being optional—are in fact an automatic consequence of this very process of quantization. We shall succeed in correctly quantizing a large number (a dense set) of Hamiltonians. In that sense the procedures to be described offer a fully satisfactory solution to the problem of providing an intrinsic and coordinate-free prescription for quantization which agrees with known quantum mechanical results.

Let us begin with a brief overview of just what it is about classical and quantum mechanics that is similar, as well as what is different. For ease of exposition we shall assume that our phase space essentially admits global coordinates. For pedagogical reasons we shall focus on just a single degree of freedom.

Symplectic manifolds and classical mechanics

For a single degree of freedom, phase space is a two-dimensional manifold M equipped with a *symplectic form* ω , which is a closed, nondegenerate two form. As a closed form it is locally exact, that is $\omega = d\theta$, where θ is a one form and d denotes the operation of exterior derivative. This one form is not unique and it follows from the fact that $d^2 = 0$ that we can replace θ by $\theta + dG$, where G is a zero form or scalar function. For a general G it follows that

$$d(\theta + dG) = d\theta + d^2G = d\theta = \omega .$$

In addition we introduce a scalar function H on the manifold M which takes on a unique value at each point $x \in M$. Now consider a path $x(t)$, $0 \equiv t' \leq t \leq t'' \equiv T$, in the manifold M parameterized by the variable t (“time”), so that θ , G , and H all become functions of t through their dependence on the phase-space points along the path. Thus we can construct an action integral according to

$$I = \int (\theta + dG - H dt) = \int (\theta - H dt) + G'' - G' ,$$

and extremal variations, which hold both end points $x(T)$ and $x(0)$ fixed, lead to the sought for classical trajectory through the manifold. The associated equations of motion do not involve G . In that sense there is an *equivalence class* of actions all of which lead to the same classical equations of motion and thus the same classical trajectory.

We may offer a more familiar formulation with the introduction of canonical coordinates. The Darboux theorem assures us that charts of local canonical coordinates, say p and q , exist for which $\theta = p dq$, $\omega = dp \wedge dq$, $G = G(p, q)$, and $H = H(p, q)$. In these terms

$$I = \int [p dq + dG(p, q) - H(p, q) dt] = \int [p dq - H(p, q) dt] + G(p'', q'') - G(p', q'),$$

where $p'', q'' \equiv p(T), q(T)$ and $p', q' \equiv p(0), q(0)$. Insisting that the vanishing of the first-order variation holding the end points p'', q'' and p', q' fixed characterizes the classical trajectory and leads to the usual Hamiltonian equations of motion, namely

$$\dot{q} = \frac{\partial H(p, q)}{\partial p}, \quad \dot{p} = -\frac{\partial H(p, q)}{\partial q}.$$

These equations of motion are independent of G , and thus we have an equivalence class of actions all of which lead to the same classical equations of motion.¹

¹We have deliberately derived the Hamiltonian equations of motion holding *both* p and q fixed at *both* the initial and the final times. This is normally an *overspecification* of the data, and there will in general be *no* solution to the equations of motion save for the lucky situation that p'', q'' just happens to lie on the future evolution of the initial phase space point p', q' . Even if we overspecify the end point data, we certainly know how to select those subcases that will generally admit solutions by focussing on the unique solution that follows from choosing just p', q' . We are willing to pay the price of having no solution, in general, in order that the action principle will be invariant under the addition of an arbitrary *phase-space* function $G(p, q)$ in the integrand; there is no alternative if the action principle is to be invariant under canonical coordinate transformations. For example, if $G(p, q) = -\frac{1}{2}pq$, it is clearly necessary to hold the variations of both p and q fixed at both boundaries in order to derive Hamilton's equations of motion.

There is some ambiguity on this point in the literature [1], which is no doubt prompted by the fact that the phase-space path integral, as it is usually formulated (in the Schrödinger representation), holds just two end point values fixed, namely q'' and q' , rather than the four end point values necessary to achieve full invariance of the action principle

New canonical coordinates, say \bar{p} and \bar{q} , are invariably related to the original canonical coordinates through the one form

$$\bar{p} d\bar{q} = p dq + dF(\bar{q}, q)$$

for some function F . It follows, therefore, that for a general G and F , one may always find a function \bar{G} such that

$$I = \int [\bar{p} d\bar{q} + d\bar{G}(\bar{p}, \bar{q}) - \bar{H}(\bar{p}, \bar{q}) dt] = \int [\bar{p} d\bar{q} - \bar{H}(\bar{p}, \bar{q}) dt] + \bar{G}(\bar{p}'', \bar{q}'') - \bar{G}(\bar{p}', \bar{q}'),$$

where $\bar{H}(\bar{p}, \bar{q}) = H(p, q)$. Extremal variation of this version of the action holding the end points fixed as before leads to Hamilton's equations expressed in the form

$$\dot{\bar{q}} = \frac{\partial \bar{H}(\bar{p}, \bar{q})}{\partial \bar{p}}, \quad \dot{\bar{p}} = -\frac{\partial \bar{H}(\bar{p}, \bar{q})}{\partial \bar{q}}.$$

This form invariance of Hamilton's equations among general canonical coordinates is what distinguishes this family of coordinate systems in the first place. Such form invariance is the clue that there is, after all, something of a geometrical nature underlying this structure, and that geometry is in fact the symplectic geometry briefly discussed above.

It is noteworthy that a given Hamiltonian expressed, say, by $H(p, q) = \frac{1}{2}(p^2 + q^2) + q^4$, in one set of canonical coordinates, could, in a new set of canonical coordinates, be expressed simply as $\bar{H}(\bar{p}, \bar{q}) = \bar{p}$. If we interpret the first expression as corresponding physically to an anharmonic oscillator, a natural question that arises is how is one to "read" out of the second expression that one is dealing with an anharmonic oscillator. This important question will figure significantly in our study!

Old quantum theory

In the old quantum theory one approximately quantized energy levels by the Bohr-Sommerfeld quantization scheme in which

$$\oint p dq = (n + \frac{1}{2})2\pi\hbar,$$

under general canonical coordinate transformations. As we shall see the coherent-state form of the path integral—to which we will automatically be led—does indeed specify the values of p'', q'' and p', q' consistent with the present discussion. Incidentally, in the coherent state case, the semi-classical limit leads, in general, to a *complex* solution [2] of the classical equations of motion that connect an arbitrary starting pair p', q' with a general final pair p'', q'' .

where the integral corresponds to a closed contour in phase space at a constant energy value. This integral is to be taken in some canonical coordinate system, but which one? Since two canonical coordinate systems are related by

$$\bar{p} d\bar{q} = p dq + dF(\bar{q}, q) ,$$

it follows, when M is simply connected, that

$$\oint \bar{p} d\bar{q} = \oint p dq ,$$

and so which canonical coordinates are used doesn't matter—they all give the same result. In coordinate-free language this remark holds simply because

$$\oint p dq = \int dp \wedge dq = \int d\bar{p} \wedge d\bar{q} \equiv \int \omega .$$

Thus we also learn one answer to the question posed above, namely the physics of the mathematical expression for the Hamiltonian given simply by \bar{p} is coded into the orbits and into the coordinate-invariant phase-space areas captured by $\int \omega$ for each value of the energy.

New quantum theory

The royal route to quantization, following Schrödinger for example, consists of first introducing a Hilbert space of functions $\psi(x)$, defined for $x \in \mathbb{R}$, each of which satisfies $\int |\psi(x)|^2 dx < \infty$. Next the classical phase-space variables p and q are “promoted” to operators, $p \rightarrow -i\hbar\partial/\partial x$ and $q \rightarrow x$, which act by differentiation and multiplication, respectively. More general quantities, such as the Hamiltonian, become operators according to the rule

$$H(p, q) \rightarrow \mathcal{H} = H(-i\hbar\partial/\partial x, x) ,$$

an expression that may have ordering ambiguities, which we will ignore directly, but which in some sense are part of the deeper question: In which canonical coordinate systems does such a quantization procedure work? The answer of Dirac is: “This assumption [of replacing classical canonical coordinates by corresponding operators] is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a Cartesian system of axes and not to more general curvilinear coordinates.” [3].

In other words, the correctness of the Schrödinger rule of quantization depends on using the right coordinates, namely Cartesian coordinates. It is worth emphasizing that Cartesian coordinates can only exist on a *flat space*.

An analogous feature is evident in the Feynman phase-space path integral formally given by

$$\mathcal{M} \int e^{(i/\hbar) \int_0^T [pq - H(p,q)] dt} \mathcal{D}p \mathcal{D}q .$$

Despite the appearance of this expression as being covariant under a change of canonical coordinates, it is, as commonly known, effectively undefined. One common way to give meaning to this expression is by means of a lattice formulation one manner of which is given by

$$\lim_{N \rightarrow \infty} M \int \exp\left\{\frac{i}{\hbar} \sum_0^N [p_{l+\frac{1}{2}}(q_{l+1} - q_l) - \epsilon H(p_{l+\frac{1}{2}}, \frac{1}{2}(q_{l+1} + q_l))]\right\} \Pi_0^N dp_{l+\frac{1}{2}} \Pi_1^N dq_l$$

where $\epsilon \equiv T/(N+1)$, $q_{N+1} \equiv q''$, $q_0 \equiv q'$, and $M = (2\pi\hbar)^{-(N+1)}$ is a suitable normalization factor. The indicated limit is known to exist for a large class of Hamiltonians leading to perfectly acceptable (Weyl-ordered) quantizations. However, it is clear that unlike the formal continuum path integral, this lattice formulation is *not* covariant under canonical coordinate transformations; in other words, this lattice expression will lead to the correct quantum mechanics only in a limited set of canonical coordinates, namely the Cartesian set referred to by Dirac.

This then is the dilemma that confronts us. The “new” quantization of Schrödinger, Heisenberg, and Feynman—the *correct* quantization from all experimental evidence—seems to depend on the choice of coordinates. This is clearly an unsettling state of affairs since nothing physical, like quantization, should depend on something so arbitrary as the choice of coordinates!

Geometric quantization

There are two attitudes that may be taken toward this apparent dependence of the very act of quantization on the choice of coordinates. The first view would be to acknowledge the “Cartesian character” that is seemingly part of the procedure. The second view would be to regard it as provisional and seeks to find a quantization formulation that eliminates this apparently unphysical feature of the current approaches.

The goal of eliminating the dependence on Cartesian coordinates in the standard approaches is no doubt one of the motivations for several programs

such as geometric quantization [4], deformation quantization [5], etc. In the first of these programs, for example, one finds the basic ingredients: (i) prequantization and (ii) polarization (real and complex), which define the framework, i.e, the *kinematics*, and (iii) one of several proposals to deal with *dynamics*. Despite noble efforts, it is not unfair to say that to date only a very limited class of dynamical systems can be treated in the geometric quantization program which also conform with the results of quantum mechanics.

The approach that we shall adopt takes the other point of view seriously, namely that the “Cartesian character” is not to be ignored. As we shall see when this feature is properly understood and incorporated, *a genuine geometric interpretation of quantization can be rigorously developed that agrees with the standard quantum mechanical results for a wide (dense) set of Hamiltonians.*

Coherent States

The concept of coherent states is sufficiently broad by now that there are several definitions. Our definition is in reality a rather old one [6], and a very general one at that, so general that it incorporates essentially all other definitions. We start with a label space \mathcal{L} , which may often be identified with the classical phase space M , and a continuous map from points in the label space to (nonzero) vectors in a Hilbert space \mathbb{H} (see below). For concreteness, let us choose the label space as the phase space for a single degree of freedom. Then each point in M may be labelled by canonical coordinates (p, q) , and we use that very pair to identify the coherent state vector itself: $|p, q\rangle$ or $\Phi[p, q]$. If we choose a different set of canonical coordinates, say (\bar{p}, \bar{q}) to identify the *same* point in M , then we associate the new coordinates to the *same* vector $|\bar{p}, \bar{q}\rangle \equiv |p, q\rangle$, or even better $\bar{\Phi}[\bar{p}, \bar{q}] \equiv \Phi[p, q]$. Although it is by no means necessary, we shall specialize to coherent states that are *unit vectors*, $\langle p, q | p, q \rangle = 1 = (\Phi[p, q], \Phi[p, q])$, for all points $(p, q) \in M$. We place only two requirements on this map from M into \mathbb{H} :

(1) *continuity*, which can be stated as joint continuity in both arguments of the coherent state overlap $\mathcal{K}(p'', q''; p', q') \equiv \langle p'', q'' | p', q' \rangle$; and

(2) *resolution of unity*, for which a positive measure μ exists such that

$$\mathbb{1} \equiv \int |p, q\rangle \langle p, q| d\mu(p, q) ,$$

where $\mathbb{1}$ is the unit operator. This last equation may be understood as

$$\begin{aligned} \langle \phi | \psi \rangle &= \int \langle \phi | p, q \rangle \langle p, q | \psi \rangle d\mu(p, q) , \\ \langle p'', q'' | \psi \rangle &\equiv \psi(p'', q'') = \int \mathcal{K}(p'', q''; p, q) \psi(p, q) d\mu(p, q) , \\ \mathcal{K}(p'', q''; p', q') &= \int \mathcal{K}(p'', q''; p, q) \mathcal{K}(p, q; p', q') d\mu(p, q) . \end{aligned}$$

Each successive relation has been obtained from the previous one by specialization of the vectors involved. The last relation, in conjunction with the fact that $\mathcal{K}(p'', q''; p', q')^* = \mathcal{K}(p', q'; p'', q'')$, implies that \mathcal{K} is the kernel of a *projection operator* onto a proper subspace of $L^2(\mathbb{R}^2, d\mu)$ composed of bounded, continuous functions that comprise the Hilbert space of interest. The fact that the representatives are bounded follows from our choice of coherent states that are all unit vectors. Although no group need be involved in our definition of coherent states, it is evident that when a group is present simplifications may occur. This is true for the canonical coherent states to which we now specialize.

With Q and P self adjoint and irreducible, and $[Q, P] = i\hbar$, the canonical coherent states defined with help of the unitary Weyl group operators for all $(p, q) \in \mathbb{R}^2 \equiv M$ by

$$|p, q\rangle = e^{-iqP/\hbar} e^{ipQ/\hbar} |\eta\rangle , \quad \langle \eta | \eta \rangle = 1$$

are a standard example for which $d\mu(p, q) = dp dq / 2\pi\hbar$. The resolution of unity holds in this case for *any* normalized fiducial vector $|\eta\rangle$. However, a useful specialization occurs if we insist that $(\Omega Q + iP)|\eta\rangle = 0$, $\Omega > 0$, leading to the ground state of an harmonic oscillator. In that case

$$\langle p', q' | p, q \rangle = \exp\{ (i/2\hbar)(p' + p)(q' - q) - (1/4\hbar)[\Omega^{-1}(p' - p)^2 + \Omega(q' - q)^2] \} .$$

We note in passing that two more familiar resolutions of unity may be obtained as limits. In particular,

$$\begin{aligned} \int \lim_{\Omega \rightarrow \infty} |p, q\rangle \langle p, q| dp dq / 2\pi\hbar &= \int |q\rangle \langle q| dq = \mathbb{1} , \\ \int \lim_{\Omega \rightarrow 0} |p, q\rangle \langle p, q| dp dq / 2\pi\hbar &= \int |p\rangle \langle p| dp = \mathbb{1} , \end{aligned}$$

where the formal vectors $|q\rangle$ satisfy $Q|q\rangle = q|q\rangle$ and $\langle q'|q\rangle = \delta(q' - q)$, and correspondingly for $|p\rangle$.

The normalized canonical coherent states $|p, q\rangle$ that follow from the condition $(\Omega Q + iP)|\eta\rangle = 0$ are in fact analytic functions of the combination $\Omega q + ip$ apart from a (normalizing) prefactor. When that prefactor is removed from the vectors and put into the integration measure, one is led directly to the Segal-Bargmann Hilbert space representation by holomorphic functions.

Symbols

Generally, and with the notation $\langle(\cdot)\rangle \equiv \langle\eta|(\cdot)|\eta\rangle$, it follows from the commutation relations that $\langle p, q|P|p, q\rangle = p + \langle P\rangle$ and $\langle p, q|Q|p, q\rangle = q + \langle Q\rangle$; if $\langle p, q|P|p, q\rangle = p$ and $\langle p, q|Q|p, q\rangle = q$ we say that the fiducial vector is *physically centered*. Observe that the labels of the coherent state vectors are *not eigenvalues* but *expectation values*; thus there is no contradiction in specifying both p and q simultaneously.

For a general operator $\mathcal{H}(P, Q)$ we introduce the *upper* symbol

$$\begin{aligned} H(p, q) &\equiv \langle p, q|\mathcal{H}(P, Q)|p, q\rangle \\ &= \langle\eta|\mathcal{H}(P + p, Q + q)|\eta\rangle \\ &= \mathcal{H}(p, q) + \mathcal{O}(\hbar; p, q) , \end{aligned}$$

and, when it exists, the *lower* symbol $h(p, q)$ implicitly defined through the relation

$$\mathcal{H} = \int h(p, q) |p, q\rangle\langle p, q| dp dq / 2\pi\hbar .$$

We note that for the harmonic oscillator fiducial vector lower symbols exist for a dense set of operators, and generally $H(p, q) - h(p, q) \simeq \mathcal{O}(\hbar)$. The association of an operator \mathcal{H} with the function $h(p, q)$ is an example of what goes under the name of Toeplitz quantization today [7].

Differentials

Several differential expressions are already implicitly contained within the coherent states. The first is the canonical one form

$$\theta \equiv i\hbar \langle |d| \rangle = i\hbar(\Phi, d\Phi) = i\hbar \Sigma_n \phi_n^* d\phi_n$$

written in coordinate-free notation, or alternatively,

$$\theta = i\hbar \langle p, q | d | p, q \rangle = p dq + \langle P \rangle dq - \langle Q \rangle dp = p dq$$

using canonical coordinates, and where we have ended with a physically centered fiducial vector. In coordinate-free notation it follows that

$$\omega \equiv d\theta = i\hbar \Sigma_n d\phi_n^* \wedge d\phi_n ,$$

and in canonical coordinates that

$$\omega = dp \wedge dq = d\bar{p} \wedge d\bar{q} ,$$

along with $d\omega = 0$ which follows directly. A useful Riemannian metric is given first in coordinate-free notation by

$$\begin{aligned} d\sigma^2 &\equiv 2\hbar^2 [\|d| \rangle\|^2 - |\langle |d| \rangle|^2] \\ &= 2\hbar^2 \Sigma_{n,m} d\phi_n^* (\delta_{nm} - \phi_n \phi_m^*) d\phi_m , \end{aligned}$$

and second in canonical coordinates by

$$\begin{aligned} d\sigma^2(p, q) &= \hbar(dp^2 + dq^2) , \quad (\Omega = 1) , \\ d\sigma^2(\bar{p}, \bar{q}) &= \hbar[A(\bar{p}, \bar{q})d\bar{p}^2 + B(\bar{p}, \bar{q})d\bar{p}d\bar{q} + C(\bar{p}, \bar{q})d\bar{q}^2] . \end{aligned}$$

In the next to the last line the line element is expressed in the Cartesian form it takes for a Gaussian fiducial vector, while in the last line is the expression of the flat metric in general canonical coordinates.

Canonical and unitary transformations

In classical mechanics canonical transformations may either be viewed as passive or active. Passive transformations leave the point in phase space fixed but change the coordinates by which it is described; active transformations describe a flow of points in phase space against a fixed coordinate system. Perhaps the best known example of an active transformation is the continuous unfolding in time of a dynamical evolution. In quantum mechanics unitary transformations are presumed to play the role that canonical transformations play in the classical theory [8]. If $p \rightarrow P$ and $q \rightarrow Q$, then it follows that $\bar{p} = (p+q)/\sqrt{2} \rightarrow (P+Q)/\sqrt{2} \equiv \bar{P}$ and $\bar{q} = (q-p)/\sqrt{2} \rightarrow (Q-P)/\sqrt{2} \equiv \bar{Q}$,

and moreover there exists a unitary operator U such that $\bar{P} = U^\dagger P U$ and $\bar{Q} = U^\dagger Q U$. Consider instead the classical canonical transformation $\tilde{p} \equiv (p^2 + q^2)/2 \rightarrow \tilde{P}$ and $\tilde{q} \equiv \tan^{-1}(q/p) \rightarrow \tilde{Q}$. As basically a transformation to polar coordinates this canonical transformation is well defined except at the single point $p = q = 0$. However, the associated quantum operators in this case cannot be connected by a unitary transformation to the original operators P and Q (because $\tilde{P} \geq 0$ and the spectrum of an operator is preserved under a unitary transformation). Thus some passive canonical transformations have images in unitary transformations while others definitely do not.

Using coherent states it is possible to completely disconnect canonical transformations and unitary transformations. Consider the transformations of the upper and lower symbols in the following example:

$$\begin{aligned} \frac{1}{2}(p^2 + q^2) &= \langle p, q | \frac{1}{2}(P^2 + Q^2 - \hbar) | p, q \rangle \\ &= \langle \tilde{p}, \tilde{q} | \frac{1}{2}(P^2 + Q^2 - \hbar) | \tilde{p}, \tilde{q} \rangle = \tilde{p} , \\ \frac{1}{2}(P^2 + Q^2 + \hbar) &= \int \frac{1}{2}(p^2 + q^2) | p, q \rangle \langle p, q | dp dq / 2\pi\hbar \\ &= \int \tilde{p} | \tilde{p}, \tilde{q} \rangle \langle \tilde{p}, \tilde{q} | d\tilde{p} d\tilde{q} / 2\pi\hbar . \end{aligned}$$

Observe in this example how the *operators* and *coherent state vectors* have remained *completely fixed* as the coordinates have passed from (p, q) to (\tilde{p}, \tilde{q}) . Of course, one may also introduce separate and arbitrary unitary transformations of the operators and vectors, e.g. $P \rightarrow V P V^\dagger$, and $| p, q \rangle \rightarrow V | p, q \rangle$, etc., which have the property of preserving inner products.

The Shadow Metric

A Flat Metric is Already Present

The form of the metric $d\sigma^2$ was given earlier for a special (harmonic oscillator) fiducial vector. If instead we consider a *general fiducial vector* $|\eta\rangle$, then it follows that

$$d\sigma^2 = \langle (\Delta Q)^2 \rangle dp^2 + \langle \Delta P \Delta Q + \Delta Q \Delta P \rangle dp dq + \langle (\Delta P)^2 \rangle dq^2 ,$$

which shows itself to be always *flat*; thus this is a property of the *Weyl group* and not of the fiducial vector. Here $\Delta P \equiv P - \langle P \rangle$, etc. Unlike the symplectic

form or the Hamiltonian, for example, the metric is typically $O(\hbar)$ and thus it is essentially nonclassical.

Indeed, any quantization scheme in which the Weyl operators and Hilbert space vectors exist leads to the metric $d\sigma^2$, whether it is intentional or not. Such quantization schemes may not explicitly *use* the metric, but it is nevertheless *there*. Moreover, any proposed quantization scheme that vigorously avoids introducing a metric must come to terms with how a flat space appears, as it must, if such a quantization scheme ultimately involves Weyl operators and Hilbert space vectors.

Cartesian Coordinates are Important

We assert that physics actually resides in Cartesian coordinates, and more generally in the coordinate form of the metric. Suppose $d\sigma^2 = \hbar(dp^2 + dq^2)$, then it follows that $H(p, q) = \frac{1}{2}(p^2 + q^2)$ implies, as before, that $\mathcal{H} = \frac{1}{2}(P^2 + Q^2 - \hbar)$. On the other hand, if instead $d\sigma^2 = \hbar[(2\tilde{p})^{-1}d\tilde{p}^2 + (2\tilde{p})d\tilde{q}^2]$, then $\tilde{H}(\tilde{p}, \tilde{q}) = \tilde{p}$ implies that $\mathcal{H} = \frac{1}{2}(P^2 + Q^2 - \hbar)$. In other words, *the physical meaning of the coordinatized mathematical expression for some classical quantity is coded into the coordinate form of the metric!* This remark is already true at the classical level, namely one needs a flat “shadow” metric (sometimes called a “secret” metric) on the classical phase space, or at least on a copy of it, so that one can ascribe physical meaning to the coordinatized mathematical expressions for one or another classical quantity. If the flat shadow metric is expressed in Cartesian coordinates, then one may interpret an expression such as $\frac{1}{2}(p^2 + q^2) + q^4$ as truly representing a physical, quartic anharmonic oscillator; if the flat shadow metric is *not* expressed in Cartesian coordinates, then no such physical interpretation of such a mathematical expression is justified.

Although we have originally introduced the phase-space metric in the quantum theory via its construction in terms of coherent states, we now see that we can alternatively view the phase-space metric (modulo a coefficient \hbar) as an auxiliary classical expression that provides physical meaning for coordinatized expressions of the classical theory.

Quantization and Continuous-time Regularization

It should be self evident that quantization relates to physical systems inasmuch as the quantization of a particular Hamiltonian is designed to generate the spectrum appropriate to that physical system. Consider again the formal phase-space path integral given by

$$\mathcal{M} \int e^{(i/\hbar) \int_0^T [p\dot{q} + \dot{G}(p,q) - h(p,q)] dt} \mathcal{D}p \mathcal{D}q .$$

We have already stressed that this expression is not mathematically defined, and now we emphasize that in fact it has no physics as well because there is no way of telling to which physical system the coordinatized expression for the Hamiltonian corresponds. In short, the formal phase-space path integral expression has neither mathematical nor physical meaning as it stands!

We will remedy this situation in a moment, but there is one “toy” analog worth introducing initially. Consider the conditionally convergent integral that is given a definition through the introduction of a regularization and its removal as in the expression

$$\int_{-\infty}^{\infty} e^{iy^2/2} dy \equiv \lim_{\nu \rightarrow \infty} \int_{-\infty}^{\infty} e^{iy^2/2 - y^2/2\nu} dy = \sqrt{2\pi i} .$$

Other regularizations may lead to the same answer, or in fact they may lead to different results; the physical situation should be invoked to choose the relevant one.

Now let us introduce a continuous-time convergence factor into the formal phase-space path integral in the form

$$\begin{aligned} & \lim_{\nu \rightarrow \infty} \mathcal{M}_\nu \int e^{(i/\hbar) \int_0^T [p\dot{q} + \dot{G}(p,q) - h(p,q)] dt} e^{-(1/2\nu) \int_0^T (\dot{p}^2 + \dot{q}^2) dt} \mathcal{D}p \mathcal{D}q \\ &= \lim_{\nu \rightarrow \infty} 2\pi\hbar e^{\hbar\nu T/2} \int e^{(i/\hbar) \int_0^T [p dq + dG(p,q) - h(p,q) dt]} d\mu_W^\nu(p, q) . \end{aligned}$$

In the first line we have formally stated the form of the regularization, while in the second line appears the proper mathematical statement it assumes after some minor rearrangement. The measure μ_W^ν is a two-dimensional Wiener measure expressed in Cartesian coordinates on the plane as signified by the metric $dp^2 + dq^2$ that appears in the first line in the regularization

factor. *Here enters explicitly the very shadow metric itself, used to give physical meaning to the coordinatized form of the Hamiltonian, and which now additionally underpins a rigorous regularization for the path integral!* As Brownian motion paths, with diffusion constant ν , almost all paths are continuous but nowhere differentiable. Thus the initial term $\int p dq$ needs to be defined as a *stochastic integral* and we choose to do so in the Stratonovich form, namely as $\lim \Sigma \frac{1}{2}(p_{l+1} + p_l)(q_{l+1} - q_l)$, where $q_l \equiv q(l\epsilon)$, etc., and the limit refers to $\epsilon \rightarrow 0$ [9]. This prescription is generally different from that of Itô, namely $\lim \Sigma p_l(q_{l+1} - q_l)$, due to the unbounded variation of the Wiener paths involved. Observe, in the second line above, for each $0 < \nu < \infty$, that *no mathematical ambiguities remain*, i.e., the expression is completely well defined. As we note below not only does the limit exist but it also provides the correct solution to the Schrödinger equation for a dense set of Hamiltonian operators.

The continuous-time regularization, as well as its Wiener measure counterpart, involves pinning the paths $p(t), q(t)$ at $t = T$ and at $t = 0$ so that $(p'', q'') = (p(T), q(T))$ and $(p', q') = (p(0), q(0))$. This leads to an expression of the form $K(p'', q'', T; p', q', 0)$, which may be shown to be

$$\begin{aligned} K(p'', q'', T; p', q', 0) &\equiv \langle p'', q'' | e^{-i\mathcal{H}T/\hbar} | p', q' \rangle , \\ |p, q\rangle &\equiv e^{-iG(p,q)/\hbar} e^{-iqP/\hbar} e^{ipQ/\hbar} |\eta\rangle , \quad (Q + iP)|\eta\rangle = 0 , \\ \mathcal{H} &\equiv \int h(p, q) |p, q\rangle \langle p, q| dp dq / 2\pi\hbar . \end{aligned}$$

In brief, the regularization chosen *automatically* leads to a coherent state representation, and, in addition, it *selects* the Hamiltonian operator determined by the lower symbol. There are three technical requirements for this representation to hold [10]:

$$\begin{aligned} (1) \quad & \int h^2(p, q) e^{-A(p^2+q^2)} dp dq < \infty , \quad \text{for all } A > 0 , \\ (2) \quad & \int h^4(p, q) e^{-B(p^2+q^2)} dp dq < \infty , \quad \text{for some } B < \frac{1}{2} , \\ (3) \quad & \mathcal{H} \text{ is e.s.a. on } D = \{\Sigma_0^N a_n |n\rangle : a_n \in \mathbf{C}, N < \infty\} , \end{aligned}$$

where the orthonormal states $|n\rangle \equiv (1/\sqrt{n!})[(Q - iP)/\sqrt{2\hbar}]^n |\eta\rangle$, $n \geq 0$. Thus this representation includes (but is not limited to) *all Hamiltonians that are Hermitian, semibounded polynomials of the basic operators P and*

Q . We note that G generally serves as an unimportant gauge; however, if the topology of M is not simply connected then G contains the Aharanov-Bohm phase [11]. Observe that the propagator formula also has an *analog physical system*, namely a two-dimensional particle moving on a flat plane in the presence of a constant magnetic field perpendicular to the plane. The limit in which the mass of the particle goes to zero projects the system onto the first Landau level.

The point of using the Stratonovich prescription for stochastic integrals is that the ordinary rules of calculus still apply [9]. Thus the rule for a canonical transformation given earlier, namely $\bar{p} d\bar{q} = p dq + dF(\bar{q}, q)$, still applies to Brownian motion paths. Consequently, just as in the classical case a function $\bar{G}(\bar{p}, \bar{q})$ exists so that after such a canonical coordinate transformation

$$\begin{aligned} \bar{K}(\bar{p}'', \bar{q}'', T; \bar{p}', \bar{q}', 0) &= \langle \bar{p}'', \bar{q}'' | e^{-i\mathcal{H}T/\hbar} | \bar{p}', \bar{q}' \rangle \\ &= \lim_{\nu \rightarrow \infty} 2\pi\hbar e^{\hbar\nu T/2} \int e^{(i/\hbar) \int_0^T [\bar{p} d\bar{q} + d\bar{G}(\bar{p}, \bar{q}) - \bar{h}(\bar{p}, \bar{q}) dt]} d\bar{\mu}_W^\nu(\bar{p}, \bar{q}) . \end{aligned}$$

Here $\bar{\mu}_W^\nu$ denotes two-dimensional Wiener measure on the flat plane expressed in general canonical coordinates.

Coordinate-free formulation

The covariant transformation of the propagator indicated above implies that a coordinate-free representation exists. We first introduce Brownian motion as a map $\rho(t; 0) : M \times M \rightarrow \mathbb{R}^+$, $t > 0$, with $\lim_{t \rightarrow 0} \rho = \delta$, $\partial\rho/\partial t = (\nu/2)\Delta\rho$, and finally $\rho(t; 0) = \int d\mu_W^\nu$ which defines a coordinate-free Wiener measure. Next we let $\phi : M \rightarrow \mathbf{C}$, $\mathcal{K} : M \times M \rightarrow \mathbf{C}$, $\phi \in (\mathcal{K})L^2(M, \omega)$, and $\phi = \mathcal{K}\phi$ (N.B. \mathcal{K} is an analog of “polarization”). Quantum dynamics comes from $i\hbar\partial\phi/\partial t = \mathcal{H}\phi$, where $\mathcal{H} = \mathcal{K}h\mathcal{K}$ (N.B. this relation has the effect of “preserving polarization”); also we introduce $K(T; 0) : M \times M \rightarrow \mathbf{C}$, so that $\phi(T) = K(T; 0)\phi(0)$. The construction of the reproducing kernel \mathcal{K} and the propagator K reads

$$\begin{aligned} \mathcal{K} &\equiv \lim_{\nu \rightarrow \infty} 2\pi\hbar e^{\hbar\nu T/2} \int e^{(i/\hbar) \int (\theta + dG)} d\mu_W^\nu = \lim_{T \rightarrow 0} K(T; 0) , \\ K(T; 0) &\equiv \lim_{\nu \rightarrow \infty} 2\pi\hbar e^{\hbar\nu T/2} \int e^{(i/\hbar) \int (\theta + dG - \hbar dt)} d\mu_W^\nu . \end{aligned}$$

Observe again how a flat metric has been used for the Brownian motion; in our view it is this flat (phase) space that underlies Dirac’s remark related to canonical quantization quoted above.

Alternative continuous-time regularizations

Our introduction of Brownian motion on a *flat* two-dimensional phase space has led to canonical quantization, namely one involving the Heisenberg operators P and Q . If instead we choose to regularize on a phase space taken as a two-dimensional *spherical* surface of radius R , where $R^2 \equiv s = \hbar/2, \hbar, 3\hbar/2, \dots$, then such a Brownian motion regularization automatically leads to a quantization in which the kinematical operators are the spin operators S_1, S_2 , and S_3 such that $\Sigma S_j^2 = s(s+1)\hbar^2$, i.e., the generators of the $SU(2)$ group [10]. In like manner, if we introduce a Brownian motion regularization on a two-dimensional *pseudo-sphere* of constant negative curvature, then the kinematical operators that automatically emerge are the generators of the affine (“ $ax + b$ ”) group, a subgroup of $SU(1,1)$ [12]. The three examples given here exhaust the simply connected spaces of constant curvature in two dimensions; they also have the property that the metric assumed for the Brownian motion regularization coincides with the metric that follows from the so-derived coherent states.

Summarizing, *the geometry of the regularization that supports the Brownian motion actually determines the nature of the kinematical operators in the quantization!*

Regularization on a General 2-D Surface

Finally, we note that the present kind of quantization can be extended to a general two-dimensional surface without symmetry and with an arbitrary number of handles. We only quote the result for the propagator. Let ξ^j , $j = 1, 2$, denote the two coordinates, $g_{jk}(\xi)$ the metric, $a_j(\xi)$ a two-vector, $f_{jk}(\xi) = \partial_j a_k(\xi) - \partial_k a_j(\xi)$ its curl, and $h(\xi)$ the classical Hamiltonian. Then the propagator is defined by [13]

$$\begin{aligned} \langle \xi'', T | \xi', 0 \rangle &= \langle \xi'' | e^{-i\mathcal{H}T/\hbar} | \xi' \rangle \\ &= \lim_{\nu \rightarrow \infty} \mathcal{M}_\nu \int \exp\{ (i/\hbar) \int [a_j(\xi) \dot{\xi}^j - h(\xi)] dt \} \\ &\quad \times \exp\{ -(1/2\nu) \int g_{jk}(\xi) \dot{\xi}^j \dot{\xi}^k dt \} \\ &\quad \times \exp\{ (\hbar\nu/4) \int \sqrt{g(\xi)} \epsilon^{jk} f_{jk}(\xi) dt \} \Pi_t \sqrt{g(\xi)} d\xi^1 d\xi^2 . \end{aligned}$$

Observe, in this general setting, that the phase-space metric tensor $g_{jk}(\xi)$ is one of the necessary *inputs* to the process of quantization under discussion. From this viewpoint the phase-space metric induced by the coherent states is regarded as a derived quantity, and in the general situation the two metrics may well differ. For a compact manifold M it is necessary that $\int f_{jk}(\xi) d\xi^j \wedge d\xi^k = 4\pi\hbar n$, $n = 1, 2, 3, \dots$. In this case the Hilbert space dimension $D = n + 1 - \bar{g}$, where \bar{g} is the number of handles in the space (genus). Here the states $|\xi\rangle$ are coherent states that satisfy

$$\begin{aligned}\mathbb{1} &= \int |\xi\rangle\langle\xi| \sqrt{g(\xi)} d\xi^1 d\xi^2, \\ \mathcal{H} &= \int h(\xi) |\xi\rangle\langle\xi| \sqrt{g(\xi)} d\xi^1 d\xi^2.\end{aligned}$$

Observe that although the states $|\xi\rangle$ are coherent states in the sense of this article, there is generally *no transitive group* with which they may be defined. The propagator expression above is manifestly covariant under arbitrary coordinate transformations, and a gauge transformation of the vector a introduces a gauge-like contribution that does not appear in the field f . Finally—and contrary to general wisdom—we note that the weighting in the case of a general geometry is *nonuniform* in the sense that the symplectic form $\omega = f_{jk} d\xi^j \wedge d\xi^k / 2$ is generally *not* proportional to the volume element $\sqrt{g(\xi)} d\xi^1 d\xi^2$ needed in the resolution of unity and hence in the path integral construction.

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